Journal of Engineering Mathematics **45:** 105–116, 2003.

Homotopy analysis of nonlinear progressive waves in deep water

SHI-JUN LIAO[1] and KWOK FAI CHEUNG[2]
[1]*School of Naval Architecture and Ocean Engineering, Shanghai Jiao Tong University, Shanghai 200030, China; Email: sjliao@mail.sjtu.edu.cn*
[2]*Department of Ocean and Resources Engineering, University of Hawaii at Manoa, Honolulu, Hawaii 96822, USA; Email: cheung@oe.soest.hawaii.edu*

Received 22 January 2002; accepted in revised form 7 July 2002

Abstract. This paper describes the application of a recently developed analytic approach known as the homotopy analysis method to derive a solution for the classical problem of nonlinear progressive waves in deep water. The method is based on a continuous variation from an initial trial to the *exact* solution. A Maclaurin series expansion provides a successive approximation of the solution through repeated application of a differential operator with the initial trial as the first term. This approach does not require the use of perturbation parameters and the solution series converges rapidly with the number of terms. In the framework of this approach, a new technique to apply the Padé expansion is implemented to further improve the convergence. As a result, the calculated phase speed at the 20th-order approximation of the solution agrees well with previous perturbation solutions of much higher orders and reproduces the well-known characteristics of being a non-monotonic function of wave steepness near the limiting condition.

Key words: deep water, homotopy analysis method, nonlinear, Padé expansion, progressive waves

1. Introduction

Consider a train of progressive gravity waves moving at a phase speed C on the surface of infinitely deep water. The two-dimensional boundary-value problem is defined with a coordinate system (x, y) fixed to the waves. The x-axis is positive in the direction of wave propagation and the y-axis points vertically upward from the still-water level. The problem is steady and is periodic in the x-variable. The fluid is assumed to be inviscid, incompressible and without surface tension. The fluid motion can be described by a velocity potential ϕ satisfying the Laplace equation

$$\nabla^2\phi(x, y) = 0 \qquad \text{for } (x, y) \in \Omega, \tag{1}$$

where the domain Ω is defined in $\{(x, y) \mid -\infty < x < +\infty, -\infty < y < \zeta(x)\}$ with ζ indicating the surface elevation. The velocity potential ϕ is subject to the free-surface boundary conditions

$$C^2\phi_{xx} + g\phi_y + \frac{1}{2}\nabla\phi\nabla(\nabla\phi\nabla\phi) - 2C\nabla\phi\nabla\phi_x = 0 \quad \text{at } y = \zeta(x), \tag{2}$$

$$\zeta(x) = \frac{1}{g}\left(C\phi_x - \frac{1}{2}\nabla\phi\nabla\phi\right) \quad \text{at } y = \zeta(x), \tag{3}$$

and the condition at deep water

$$\lim_{y \to -\infty} \frac{\partial \phi}{\partial y} = 0, \tag{4}$$

where g denotes the gravitational acceleration and the subscripts x and y denote partial derivatives in the respective directions.

Although the governing equation (1) is linear, the free-surface boundary conditions (2) and (3) are nonlinear and are defined on a surface, which is unknown *a priori*. This classical water-wave problem does not have a straightforward solution and has attracted attention from many researchers since the mid 19th century. Stokes [1] first proposed a perturbation technique for this problem and later obtained an analytic solution to the 5th order in wave amplitude [2, 3]. Thereafter, researchers have applied Stokes' perturbation approach and derived higher-order solutions [4–6]. With the use of a computer, Schwartz [7] extended Stokes' perturbation expansion to obtain a solution to the 58th order. The solution is obtained in the complex plane through a mapping function. His perturbation expansion has limited convergence and the Padé technique is employed to derive the solution at the limiting wave condition $(H/L)_{\max} = 0{\cdot}14118$, where H is the wave height and L the wavelength.

Following Schwartz [7], Longuet-Higgins [8] took the Stokes-type expansion in wave amplitude to high orders and obtained stable solutions up to the wave steepness $H/L = 0{\cdot}1411$. The results show that for a given wavelength the energy and phase speed are not monotonic functions of wave steepness. Besides, Longuet-Higgins [9, 10] investigated the stability of steady gravity waves to infinitesimal disturbances and found subharmonic modes that become unstable, when the wave height reaches a certain value, may become stable and then unstable again as the wave height continues to increase. Chen and Saffman [11] found by numerical techniques that symmetrical steady gravity waves of large amplitudes have bifurcations at $H/L \approx 0{\cdot}13$. Additional high-order solutions based on Stokes's perturbation approach further illustrate the nonlinear characteristics of steep gravity waves [12–15].

The present paper describes a new analytic approach to the solution of this classical water-wave problem. The solution is based on the homotopy analysis method [16], which does not require the assumption of small or large quantities as perturbation parameters. Different from other analytic techniques, this method provides a simple way to control the convergence rate and region of the approximation series. The comparison of the homotopy analysis method with other perturbation and non-perturbation techniques is discussed in detail by Liao [17]. The homotopy analysis method has been successfully applied to derive explicit analytic solutions for a number of classical nonlinear problems including Blasius' viscous-flow problem [18], the Falkner-Skan viscous flow over a semi-infinite plane [19], and the drag of a sphere over a considerably larger range of Reynolds number than that of all previous theoretical solutions [20]. These studies have verified the validity of the homotopy analysis method as a powerful analytic tool for nonlinear problems.

The homotopy analysis method is applied here for the first time to a nonlinear problem with multiple unknown variables and an *unknown* boundary. The method has several advantages over the perturbation technique in the solution of the nonlinear water-wave problem defined by Equations (1–4). It provides greater flexibility in the selection of a proper set of basis functions for the solution and provides a simple way to implement the Padé expansion to improve the convergence and accuracy of the solution series. This allows the formulation of the boundary-value problem in the physical plane in contrast to Schwartz's approach [7] and gives rise to a general solution approach for a larger class of nonlinear problems.

2. Mathematical formulation

2.1. Continuous variation

The homotopy analysis method is based on a continuous variation from an initial trial to the *exact* solution. In the water-wave problem, we construct the mappings $\phi(x, y) \to \Phi(x, y; p)$, $\zeta(x) \to \eta(x; p)$, and $C \to \Lambda(p)$ such that, as the embedding parameter p increases from 0 to 1, $\Phi(x, y; p)$, $\eta(x; p)$, and $\Lambda(p)$ vary from the initial guesses to the exact solution given by $\phi(x, y)$, $\eta(x)$, and C, respectively. To ensure this, $\Phi(x, y; p)$ satisfies the Laplace equation

$$\nabla^2\Phi(x, y; p) = 0 \quad \text{for } (x, y) \in \overline{\Omega}(p), \tag{5}$$

where the domain $\overline{\Omega}(p) = \{(x, y) \mid -\infty < x < +\infty, -\infty < y < \eta(x; p)\}$ should preserve the connectedness as p spans the interval [0, 1]. The potential Φ is subject to three boundary conditions analogous to (2), (3), and (4), given, respectively, by

$$(1 - p)\mathcal{L}[\Phi(x, y; p) - \phi_0(x, y), \Lambda(p)] = p\hbar\mathcal{N}[\Phi(x, y; p), \Lambda(p)] \quad \text{at } y = \eta(x; p), \tag{6}$$

$$(1 - p)[\eta(x; p) - \zeta_0(x)] = p\hbar\{\eta(x; p) - \mathcal{Z}[\Phi(x, y; p), \Lambda(p)]\} \quad \text{at } y = \eta(x; p), \tag{7}$$

$$\lim_{y\to-\infty} \frac{\partial\Phi(x, y; p)}{\partial y} = 0, \tag{8}$$

where $\hbar$ is a nonzero auxiliary parameter, $\phi_0(x, y)$ is an initial guess of the velocity potential satisfying (1) and (4), $\zeta_0(x, y)$ is an initial guess of the surface elevation, and $\mathcal{L}$, $\mathcal{N}$, and $\mathcal{Z}$ are differential operators determined from the free-surface boundary conditions (2) and (3).

The linear auxiliary operator $\mathcal{L}$ is second order and can be selected from a number of possible candidates. The choice of this operator will affect the convergence of the solution. Based on the two linear terms of the free-surface boundary condition (2), we select

$$\mathcal{L}[\Phi(x, y; p), \Lambda(p)] = \Lambda^2(p)\frac{\partial^2\Phi(x, y; p)}{\partial x^2} + g\frac{\partial\Phi(x, y; p)}{\partial y}. \tag{9}$$

The nonlinear operators $\mathcal{N}$ and $\mathcal{Z}$ are given, respectively, by the two free-surface boundary conditions (2) and (3) as

$$\begin{aligned} \mathcal{N}[\Phi(x, y; p), \Lambda(p)] &= \Lambda^2(p)\Phi_{xx}(x, y; p) + g\Phi_y(x, y; p) \\ &+ \frac{1}{2}\nabla\Phi(x, y; p)\nabla[\nabla\Phi(x, y; p)\nabla\Phi(x, y; p)] \\ &- 2\Lambda(p)\nabla\Phi(x, y; p)\nabla\Phi_x(x, y; p), \end{aligned} \tag{10}$$

$$\mathcal{Z}[\Phi(x, y; p), \Lambda(p)] = \frac{1}{g}[\Lambda(p)\Phi_x(x, y; p) - \frac{1}{2}\nabla\Phi(x, y; p)\nabla\Phi(x, y; p)]. \tag{11}$$

When $p = 0$, the governing equation (5) and the boundary conditions (6) to (8) give rise to the initial trial solution

$$\Phi(x, y; 0) = \phi_0(x, y), \quad \eta(x, 0) = \zeta_0(x), \quad \Lambda(0) = C_0, \tag{12}$$

where C_0 is the initial guess of the phase speed. When $p = 1$, Equations (5–8) are equivalent to (1–4) of the original boundary-value problem with

$$\Phi(x, y; 1) = \phi(x, y), \quad \eta(x, 1) = \zeta(x), \quad \Lambda(1) = C. \tag{13}$$

The boundary-value problem defined by (5–8) thus provides a continuous variation from the initial trial to the exact solution as p increases from 0 to 1.

The convergence of the solution also depends on the choice of the initial approximation. A logical choice is the solution of the linear Airy wave theory that gives

$$\phi_0(x, y) = AC_0 \exp(ky) \sin(kx), \tag{14}$$

$$\Lambda(0) = \sqrt{\frac{g}{k}} = C_0, \tag{15}$$

as the initial approximations of $\phi(x, y)$ and C, respectively, where A is a constant and $k = 2\pi/L$ is the wave number. In spite of the more obvious choice from the linear solution, we choose

$$\zeta_0(x) = 0 \tag{16}$$

as the initial guess of the surface elevation $\zeta(x)$ to simplify the subsequent formulation and the solution procedure.

2.2. Successive approximation

The solution to the nonlinear water-wave problem is determined by a successive approximation of the continuous variation. By Maclaurin series, $\Phi(x, y; p)$, $\eta(x; p)$, and $\Lambda(p)$ are expanded about the embedding parameter p to give

$$\Phi(x, y; p) \sim \phi_0(x, y) + \sum_{m=1}^{+\infty} \frac{\phi_0^{[m]}(x, y)}{m!} p^m, \tag{17}$$

$$\eta(x; p) \sim \zeta_0(x) + \sum_{m=1}^{+\infty} \frac{\zeta_0^{[m]}(x)}{m!} p^m, \qquad \Lambda(p) \sim C_0 + \sum_{m=1}^{+\infty} \frac{C_0^{[m]}}{m!} p^m, \tag{18, 19}$$

in which

$$\phi_0^{[m]}(x, y) = \left.\frac{\partial^m \Phi(x, y; p)}{\partial p^m}\right|_{p=0}, \quad \Phi^{[m]}(x, y; p) = \frac{\partial^m \Phi(x, y; p)}{\partial p^m}, \tag{20}$$

$$\zeta_0^{[m]}(x) = \left.\frac{\partial^m \eta(x; p)}{\partial p^m}\right|_{p=0}, \quad \eta^{[m]}(x; p) = \frac{\partial^m \eta(x; p)}{\partial p^m}, \tag{21}$$

$$C_0^{[m]} = \left.\frac{\mathrm{d}^m \Lambda(p)}{\mathrm{d}p^m}\right|_{p=0}, \quad \Lambda^{[m]} = \frac{\mathrm{d}^m \Lambda(p)}{\mathrm{d}p^m}. \tag{22}$$

As is evident from (13), any converging series given by the homotopy analysis method at $p = 1$ represents the exact solution. If the value of $\hbar$ is properly selected so that the above Maclaurin series are convergent at $p = 1$, we have

$$\phi(x, y) = \phi_0(x, y) + \sum_{m=1}^{+\infty} \frac{\phi_0^{[m]}(x, y)}{m!}, \tag{23}$$

$$\zeta(x) = \zeta_0(x) + \sum_{m=1}^{+\infty} \frac{\zeta_0^{[m]}(x)}{m!}, \tag{24}$$

$$C = C_0 + \sum_{m=1}^{+\infty} \frac{C_0^{[m]}}{m!}. \tag{25}$$

The above formulas provide a series solution for the exact boundary-value problem with the initial trial solution, $\phi_0(x, y)$, $\zeta_0(x)$, and C_0, as the first term. The unknown $\phi_0^{[m]}(x, y)$, $\zeta_0^{[m]}(x)$, and $C_0^{[m]}$ are determined in the order $m = 1, 2, 3, \cdots$ with governing equations and boundary conditions established in turn from (5–8).

Differentiating Equations (5) and (8) m times with respect to the embedding parameter at $p = 0$, we have the governing equation

$$\nabla^2 \phi_0^{[m]}(x, y) = 0 \quad \text{in } (x, y) \in \Omega_0 \tag{26}$$

and the condition at deep water

$$\lim_{y \to -\infty} \frac{\partial \phi_0^{[m]}(x, y)}{\partial y} = 0. \tag{27}$$

The free-surface boundary conditions (6) and (7) are satisfied at $y = \eta(x; p)$, which itself is a function of p. Thus, it holds for $\Phi(x, y; p)$ at $y = \eta(x; p)$ that

$$\frac{\mathrm{D}^m \Phi(x, y; p)}{\mathrm{D}p^m} = \left[\frac{\partial}{\partial p} + \eta^{[1]}(x; p) \frac{\partial}{\partial y} \right]^m \Phi(x, y; p), \tag{28}$$

where $\eta^{[1]}(x; p)$ is given by Equation (21). The differential operator $\mathrm{D}^m/\mathrm{D}p^m$, which contains the linear term $\partial^m/\partial p^m$, is determined from a simple procedure described in the Appendix. For simplicity, we write

$$\frac{\mathrm{D}^m \Phi(x, y; p)}{\mathrm{D}p^m} = \Phi^{[m]}(x, y; p) + \mathcal{R}_m[\Phi(x, y; p), \Lambda(p)], \tag{29}$$

where $\mathcal{R}_m$ is a nonlinear operator and $\Phi^{[m]}(x, y; p)$ is defined by Equation (20). For functions independent of y, such as $\Lambda(p)$ and $\eta(x, p)$, we simply have

$$\frac{\mathrm{D}^m \eta(x; p)}{\mathrm{D}p^m} = \frac{\partial^m \eta(x; p)}{\partial p^m} = \eta^{[m]}(x; p), \tag{30}$$

$$\frac{\mathrm{D}^m \Lambda(p)}{\mathrm{D}p^m} = \frac{\mathrm{d}^m \Lambda(p)}{\mathrm{d}p^m} = \Lambda^{[m]}(p), \tag{31}$$

which are consistent with the relations defined in (21) and (22), respectively.

Differentiating Equations (6) and (7) m times with respect to the embedding parameter at $p = 0$ we obtain the respective free-surface boundary conditions defined at $y = \zeta_0(x)$ as

$$\left. \frac{\mathrm{D}^m \mathcal{L}[\Phi(x, y; p), \Lambda(p)]}{\mathrm{D}p^m} \right|_{p=0} = m \left\{ \chi_m \frac{\mathrm{D}^{m-1} \mathcal{L}[\Phi(x, y; p), \Lambda(p)]}{\mathrm{D}p^{m-1}} + \hbar \frac{\mathrm{D}^{m-1} \mathcal{N}[\Phi(x, y; p), \Lambda(p)]}{\mathrm{D}p^{m-1}} \right\} \Bigg|_{p=0} \tag{32}$$

and

$$\zeta_0^{[m]}(x) = m\left\{(\chi_m + \hbar)\zeta_0^{[m-1]}(x) - \hbar\frac{\mathrm{D}^{m-1}\mathcal{Z}[\Phi(x,y;p),\Lambda(p)]}{\mathrm{D}p^{m-1}}\bigg|_{p=0}\right\}, \tag{33}$$

in which

$$\begin{aligned}&\frac{\mathrm{D}^m\mathcal{L}[\Phi(x,y;p),\Lambda(p)]}{\mathrm{D}p^m}\bigg|_{p=0}\\ &=\sum_{i=0}^{m}\binom{m}{i}\frac{\mathrm{D}^i[\Lambda^2(p)]}{\mathrm{D}p^i}\bigg|_{p=0}\frac{\mathrm{D}^{m-i}\Phi_{xx}(x,y;p)}{\mathrm{D}p^{m-i}}\bigg|_{p=0}+g\frac{\mathrm{D}^m\Phi_y(x,y;p)}{\mathrm{D}p^m}\bigg|_{p=0}\end{aligned} \tag{34}$$

where $\chi_1 = 0$ and $\chi_m = 1$ for $m \geq 2$. Substituting Equation (29) in (32), we have

$$C_0^2\frac{\partial^2\phi_0^{[m]}(x,y)}{\partial x^2} + g\frac{\partial^2\phi_0^{[m]}(x,y)}{\partial y} = S_m(x,y) \quad \text{at } y = \zeta_0(x), \tag{35}$$

in which

$$\begin{aligned}S_m(x,y) = &\left\{m\chi_m\frac{\mathrm{D}^{m-1}\mathcal{L}[\Phi(x,y;p),\Lambda(p)]}{\mathrm{D}p^{m-1}} + m\hbar\frac{\mathrm{D}^{m-1}\mathcal{N}[\Phi(x,y;p),\Lambda(p)]}{\mathrm{D}p^{m-1}}\right.\\ &-C_0^2\mathcal{R}_m[\Phi_{xx}(x,y;p),\Lambda(p)] - g\mathcal{R}_m[\Phi_y(x,y;p),\ \Lambda(p)]\\ &\left.-\sum_{i=1}^{m}\binom{m}{i}\frac{\mathrm{D}^i[\Lambda^2(p)]}{\mathrm{D}p^i}\frac{\mathrm{D}^{m-i}[\Phi_{xx}(x,y;p)]}{\mathrm{D}p^{m-i}}\right\}\bigg|_{p=0}.\end{aligned} \tag{36}$$

Notice that the resulting boundary conditions (33) and (35) are evaluated at the initial approximation of the surface elevation $\zeta_0(x)$ and the reason for choosing $\zeta_0(x) = 0$ is now evident.

2.3. Solution procedure

The boundary-value problem at the mth-order approximation is defined by the governing Equation (26) and the boundary conditions (27), (33) and (35). Although the embedding parameter p is used in the Maclaurin series expansions (17–19), it vanishes in the resulting boundary-value problem and its solution. The right-hand side of Equation (33) is only dependent upon terms up to the $(m-1)$th approximation. Thus, $\zeta_0^{[m]}(x)$ can be determined from (33) before solving for the mth-order approximation. Once $\zeta_0^{[m]}(x)$ is known, $S_m(x,y)$ can be evaluated and expressed in the form

$$S_m(x,y) = \sum_{n=1}^{m} b_{m,n}\sin(nkx) \quad \text{for } m \geq 1. \tag{37}$$

To avoid the secular terms in the solution of $\phi_0^{[m]}(x,y)$ from Equation (35), the following condition must hold:

$$b_{m,1} = 0 \text{ for } m \geq 1. \tag{38}$$

This provides a linear algebraic equation in the form

$$\alpha_{m,1} C_0^{[m]} + \beta_{m,1} = 0, \tag{39}$$

from which the unknown $C_0^{[m]}$ can be determined in terms of the coefficients $\alpha_{m,1}$ and $\beta_{m,1}$.

Based on the boundary condition (35) and the expression for S_m in (37), the solution of $\phi_0^{[m]}(x, y)$ has the form

$$\phi_0^{[m]}(x, y) = \sum_{n=1}^{m} a_{m,n} \exp(nky) \sin(nkx), \tag{40}$$

in which

$$a_{m,n} = \frac{b_{m,n}}{(kn)g - C_0^2 (kn)^2} \quad \text{for } 2 \le n \le m. \tag{41}$$

Notice that the coefficient $a_{m,1}$ is still unknown. To relate the solution and the wave height H, we enforce

$$\zeta_0^{[m]}(0) - \zeta_0^{[m]}(L/2) = \begin{cases} H & \text{for } m = 1 \\ 0 & \text{for } m \ge 2. \end{cases} \tag{42}$$

This relation provides a second linear algebraic equation in the form

$$\alpha_{m,2} a_{m,1} + \beta_{m,2} = 0, \tag{43}$$

for the solution of $a_{m,1}$ in terms of the coefficients $\alpha_{m,2}$ and $\beta_{m,2}$. The value of A in the initial approximation of $\phi_0(x, y)$ in Equation (14) is determined from (33) and (42) as

$$A = -\left(\frac{H}{2\hbar}\right). \tag{44}$$

The above expression suggests that $\hbar$ should be negative and nonzero.

The nonlinear water-wave problem is now reduced to the two linear algebraic equations for $C_0^{[m]}$ and $a_{m,1}$, respectively. The solutions of the two equations complete the expression for $\phi_0^{[m]}(x, y)$, as well as the mth-order approximation of the solution. The formulation can be easily adapted for symbolic computation and we obtained the analytic solution using Mathematica Version 4.1. Since symbolic computation is employed, truncation error is not a concern for the numerical results presented in this paper.

3. Results and analysis

The phase speed C, velocity potential $\phi(x, y)$, and surface elevation $\zeta(x)$ are dependent upon the wave steepness. Mathematically, all of them are also functions of the auxiliary parameter $\hbar$, which influences the convergence rate and region of the solution series (23–25). In practice, a finite number of terms are used in the solution series. The Mth-order approximation of (23–25) become

$$\phi(x, y) \approx \phi_0(x, y) + \sum_{m=1}^{M} \frac{\phi_0^{[m]}(x, y)}{m!}, \tag{45}$$

$$\zeta(x) \approx \zeta_0(x) + \sum_{m=1}^{M} \frac{\zeta_0^{[m]}(x)}{m!}, \tag{46}$$

$$C \approx C_0 + \sum_{m=1}^{M} \frac{C_0^{[m]}}{m!}. \tag{47}$$

The first few terms of the series normally provide an accurate approximation of the solution. More terms are required for convergence at the limiting wave condition depending on the choice of $\hbar$.

Most researchers focus their attention on the dispersion relation between the phase speed C and wave height H. Schwartz [7] provided the Mth-order approximation of the phase speed in the form

$$C^2 \approx \sum_{j=0}^{M} \alpha_j H^{2j}, \tag{48}$$

where α_j is constant. Schwartz applied the Padé technique to improve the convergence and obtained the solution with the maximum wave steepness $(H/L)_{\max} = 0{\cdot}14118$. In the present approach, the Mth-order approximation of the phase speed is

$$C \approx \sum_{j=0}^{M} \beta_j H^{2j}, \tag{49}$$

where β_j are coefficients. It is obvious that C^2 given by the present approach at the Mth-order approximation contains terms up to H^{4M}, whereas the highest exponent of Schwartz's formula at the same order of approximation is $2M$. The accuracy and convergency of the phase speed given by Equation (49) can be further enhanced by the following homotopy-Padé technique.

The value of $\Lambda(p)$ varies from the initial guess $C_0 = \sqrt{g/k}$ to the exact phase speed C as p increases from 0 to 1. The Maclaurin series (19) can be regarded as a power series of p. With the (2κ)th-order approximation of the solution evaluated, we can enhance the convergence by applying the $[\kappa, \kappa]$ Padé expansion to the power series

$$\Lambda(p) \approx C_0 + \sum_{m=1}^{2\kappa} \frac{C_0^{[m]}}{m!} p^m. \tag{50}$$

The resulting expansion can be organized in the form

$$\Lambda(p) \approx \frac{C_0 + \sum_{n=1}^{\kappa} B_{2\kappa,n} p^n}{1 + \sum_{n=1}^{\kappa} B_{2\kappa,\kappa+n} p^n}, \tag{51}$$

where the coefficient $B_{2\kappa,n}$ is independent of the auxiliary parameter $\bar{h}$. Setting $p = 1$ in above expression, we obtain a new approximation of the phase speed due to (13)

$$\frac{C}{C_0} \approx \frac{1 + \sum_{n=1}^{\kappa(\kappa+1)/2} \gamma_{2\kappa,n} (kH)^{2n}}{1 + \sum_{n=1}^{\kappa(\kappa+1)/2} \delta_{2\kappa,n} (kH)^{2n}}, \tag{52}$$

Table 1. Comparison of the $[\kappa, \kappa]$ homotopy-Padé approximation of C^2/C_0^2 with Schwartz [7]

H/L	Schwartz [7]	$\kappa = 6$	$\kappa = 8$	$\kappa = 10$	$\kappa = 11$
0·040	1·01592	1·01592	1·01592	1·01592	1·01592
0·070	1·04955	1·04955	1·04955	1·04955	1·04955
0·100	1·10367	1·10367	1·10367	1·10367	1·10367
0·120	1·15182	1·15190	1·15184	1·15182	1·15181
0·130	1·17820	1·17865	1·17834	1·17821	1·17821
0·135	1·18996	1·19148	1·19061	1·19003	1·19003
0·140	1·1930	1·20150	1·19833	1·19369	1·19385

where $\gamma_{2\kappa,j}$ and $\delta_{2\kappa,j}$ are coefficients. Although the power series (50) is a function of the auxiliary parameter $\hbar$, the application of the $[\kappa, \kappa]$ Padé expansion eliminates $\hbar$ from the resulting expression.

In the homotopy analysis method, the auxiliary parameter $\hbar$ controls the convergence rate and region of the approximation series. As $\hbar$ approaches 0, the convergence region enlarges at the expense of the convergence rate and increasing number of terms are needed in the approximation to maintain the same level of accuracy. The homotopy-Padé expression (52), however, does not depend on $\hbar$ and gives converging results over a considerably large convergence region at the same order of approximation. The introduction of the nonzero auxiliary parameter $\hbar$ in the formulation has only mathematical meaning here. When the traditional Padé method is applied, the resulting expression will be subject to the convergence requirements imposed by $\hbar$. Furthermore, the method requires an approximation to $\mathrm{O}(H^{2\kappa^2+2\kappa})$. In this sense, the homotopy-Padé expression (52) is to $\mathrm{O}(H^{2\kappa^2+2\kappa})$, which is considerably higher than $\mathrm{O}(H^{2\kappa})$ achieved by the Padé expansion used by Schwartz [7].

Table 1 lists the relative phase speed, C^2/C_0^2, computed at various levels of the homotopy-Padé approximation (52) and from Schwartz's perturbation solution to $\mathrm{O}(H^{116})$ [7]. For wave steepness up to $H/L = 0{\cdot}10$, the homotopy-Padé approximation converges at [6,6] and gives results identical to Schwartz's for the number of decimals considered. The computed relative phase speed at this level of approximation is to $\mathrm{O}(H^{82})$, which is lower than that considered by Schwartz. At the 20th-order approximation of the solution series, C^2 given by the [10,10] homotopy-Padé approximation is to $\mathrm{O}(H^{220})$ and converges to slightly different results in comparison to Schwartz's for wave steepness $H/L > 0{\cdot}12$. The homotopy-Padé approximation converges rapidly with the number of terms and the 20th and 22nd-order approximations of the solution give identical or very similar results over the range of wave steepness considered, indicating reasonable convergence at the 20th order and the validity of the proposed homotopy-Padé technique.

The [10,10] and [11,11] homotopy-Padé approximations of C/C_0 are compared with Longuet-Higgins's perturbation solution [8] in Table 2. The two homotopy-Padé approximations and Longuet-Higgins'results are identical for wave steepness up to $H/L = 0{\cdot}121921$, whereas the [10,10] Homotopy-Padé approximation remains convergent up to $H/L = 0{\cdot}137249$ for the number of decimals considered. The phase speeds computed by the various methods as the wave steepness approaches the limiting condition are compared in Figure 1.

Table 2. Comparison of the $[\kappa, \kappa]$ homotopy-Padé approximation of C/C_0 with Longuet-Higgins [8]

H/L	Longuet-Higgins [8]	$\kappa = 10$	$\kappa = 11$	H/L	Longuet-Higgins [8]	$\kappa = 10$	$\kappa = 11$
0	1·00000	1·00000	1·00000	0·133178	1·08904	1·08906	1·08906
0·045266	1·01016	1·01016	1·01016	0·136178	1·09184	1·09188	1·09188
0·064351	1·02065	1·02065	1·02065	0·136723	1·09222	1·09228	1·09228
0·079187	1·03143	1·03143	1·03143	0·137249	1·09255	1·09260	1·09260
0·091809	1·04247	1·04247	1·04247	0·137755	1·09275	1·09284	1·09285
0·102959	1·05366	1·05366	1·05366	0·138242	1·09290	1·09300	1·09301
0·108093	1·05926	1·05926	1·05926	0·138712	1·09295	1·09306	1·09308
0·112962	1·06482	1·06482	1·06482	0·139170	1·09291	1·09302	1·09305
0·117572	1·07029	1·07029	1·07029	0·139610	1·09279	1·09285	1·09290
0·121921	1·07558	1·07558	1·07558	0·140060	1·09258	1·09250	1·09258
0·125993	1·08059	1·08060	1·08060	0·140530	1·09240	1·09189	1·09202
0·129760	1·08516	1·08517	1·08517	0·141100	1·09230	1·09066	1·09089

Both the present and Longuet-Higgins's approaches gives the maximum phase speed at the same wave steepness $H/L = 0{\cdot}138712$ and show the phase speed is not a monotonic function of wave steepness. The homotopy–Padé approximations agree well with Longuet-Higgins's results up to $H/L = 0{\cdot}14$, but show a more rapid decrease of the phase speed toward the limiting-wave condition beyond that. The phase speed given by Schwartz [7] at $H/L = 0{\cdot}14$ is slightly lower in comparison to the other predictions.

As observed in the previous and present studies, the physics of steep gravity waves is rather complicated and different approaches produce slightly different solutions toward the limiting-wave condition. The limiting wave is unstable physically and might be mathematically as well. It would be interesting to employ the present approach to investigate the bifurcations of gravity waves for $H/L \approx 0{\cdot}13$, found numerically by Chen and Saffman [11], if higher-order approximations can be obtained in the future.

4. Conclusions

We have applied the homotopy analysis method to provide an analytic solution for the classical problem of nonlinear progressive waves in deep water and developed a new technique, namely the homotopy-Padé method, to increase the accuracy and convergence of the solution. The [10,10] homotopy-Padé approximation of C^2/C_0^2 is to $O(H^{220})$, and agrees well with Schwartz's results to $O(H^{116})$ [7]. The same approximation also gives good agreement with Longuet-Higgins's results [8] up to the wave steepness $H/L = 0{\cdot}14$ and shows that the phase speed is not a monotonic function of wave steepness. The present and the two previous approaches, however, give slightly different results toward the limiting-wave condition.

This study shows that the homotopy analysis method is applicable to a complicated nonlinear problem with two nonlinear boundary conditions defined on an unknown surface. The present approach does not involve a perturbation parameter and shows better convergence compared to other approximation techniques. The embedding parameter used in the series expansion vanishes in the resulting boundary-value problem and its solution. Most impor-

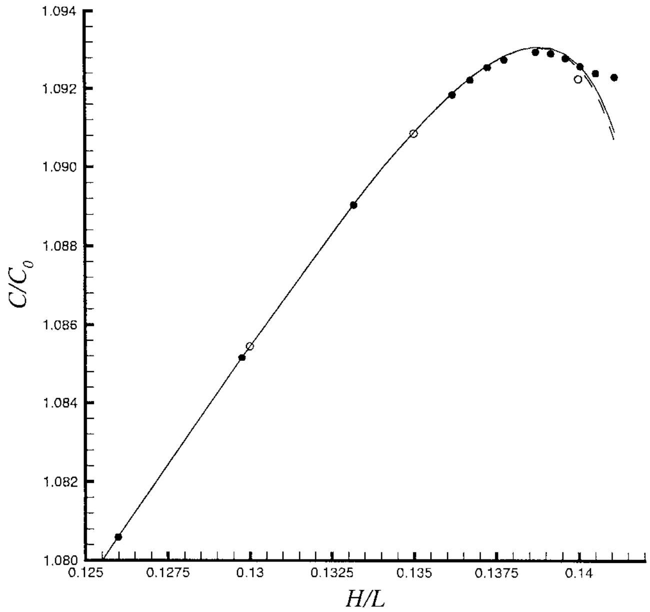

Figure 1. Phase speed *vs.* wave steepness for nonlinear progressive waves in deep water. - - - -, [10, 10] homotopy-Padé approximation; —-, [11,11] homotopy-Padé approximation; ○, Schwartz [7]; • Longuet-Higgins [8].

tantly, the application of the homotopy-Padé expansion to the solution series eliminates the auxiliary parameter used in the formulation and thus achieves a high convergence rate over a considerably large convergence region. This expansion method has potential application to a wide variety of nonlinear problems in science and engineering.

Acknowledgement

This paper is based on work funded in part by National Science Foundation of China under Grant No. 50125923. Additional support was provided by NASA Office of Earth Science under Grant No. NAG5-8748, while the first author was a visiting researcher at the University of Hawaii. We would like to thank the four anonymous reviewers for their valuable comments and suggestions.

Appendix

The operator $\mathrm{D}^m/\mathrm{D}p^m$ for $m \geq 1$ can be determined by following the procedure outlined here. The potential $\Phi(x, y; p)$ on the free surface at $y = \eta(x; p)$ can be expanded about $p = 0$ by a Taylor series to give

$$\Phi(x, y; p) = \sum_{m=0}^{+\infty} \left. \frac{\mathrm{D}^m \Phi(x, y; p)}{\mathrm{D}p^m} \right|_{p=0} \left(\frac{p^m}{m!} \right). \tag{53}$$

The potental can similarly be expanded by a Taylor series about the free surface at $y = \eta(x; 0)$ as

$$\Phi(x, y; p) = \sum_{n=0}^{+\infty} \sum_{r=0}^{+\infty} \left. \frac{\partial^n \Phi^{[r]}(x, y; p)}{\partial y^n} \right|_{p=0} \left(\frac{p^r}{n!r!} \right) [\eta(x; p) - \eta(x, 0)]^n. \tag{54}$$

Equating the two expressions for $\Phi(x, y; p)$ and invoking (12) and (18), we have

$$\sum_{m=0}^{+\infty} \left. \frac{\mathrm{D}^m \Phi(x, y; p)}{\mathrm{D} p^m} \right|_{p=0} \left(\frac{p^m}{m!} \right) = \sum_{n=0}^{+\infty} \sum_{r=0}^{+\infty} \left. \frac{\partial^n \Phi^{[r]}(x, y; p)}{\partial y^n} \right|_{p=0} \left(\frac{p^r}{n!r!} \right) \left[\sum_{s=1}^{+\infty} \left(\frac{p^s}{s!} \right) \zeta_0^{[s]}(x) \right]^n . \tag{55}$$

Expanding the right-hand side of above equation and comparing the coefficients in the same power of p, we arrive at the definition of the operator $\mathrm{D}^m/\mathrm{D}p^m$ for $m \geq 1$. This can be accomplished by symbolic computation using Mathematica Version 4.1 along with the rest of the formulation.

References

1. G.G. Stokes, On the theory of oscillation waves. *Trans. Cambridge Phil. Soc.* 8 (1849) 441–455.
2. G.G. Stokes, Supplement to a paper on the theory of oscillation waves. *Math. Phys. Papers* 1 (1880) 314–326.
3. G.G. Stokes, On the highest waves of uniform propagation. *Proc. Cambridge Phil. Soc.* 4 (1883) 361–365.
4. D.J. Korteweg and G. de Vries, On the change of form of long waves advancing in a rectangular canal and on a new type of long stationary waves. *Phil. Mag.* 39 (1895) 422–443.
5. J.V. Wehausen and E.V. Laitone, In: S. Flügge (ed.), *Handbush der Physik,* Volume 9. Berlin: Springer-Verlag (1960).
6. C.C. Mei, The nonlinear evolution of stokes waves in deep water. *J. Fluid Mech.* 47 (1971) 337–352.
7. L.W. Schwartz, Computer extension and analytic continuation of Stokes' expansion for gravity waves. *J. Fluid Mech.* 62 (1974) 553–578.
8. M.S. Longuet-Higgins, Integral properties of periodic gravity waves of finite amplitudes. *Proc. R. Soc. London* A342 (1975) 157–174.
9. M.S. Longuet-Higgins, The instability of gravity of finite amplitude in deep water (1): superharmonics. *Proc. R. Soc. London* A360 (1978) 471–488.
10. M.S. Longuet-Higgins, The instability of gravity of finite amplitude in deep water (2): subharmonics. *Proc. R. Soc. London* A360: (1978) 489–505.
11. B. Chen and P.G. Saffman, Numerical evidence for the existence of new types of gravity waves of permanent form on deep water. *Studies Appl. Math.* 62 (1980) 1–21.
12. E.D. Cokelet, Steep gravity waves in water of arbitrary uniform depth. *Phil. Trans. R. Soc. London* A286 (1977) 183–230.
13. M.S. Longuet-Higgins and E.D. Cokelet, The deformation of steep surface waves on water (i): a numerical method of computation. *Proc. R. Soc. London* A350 (1976) 1–26.
14. M.S. Longuet-Higgins and E.D. Cokelet, The deformation of steep surface waves on water (ii): growth of normal mode instabilities. *Proc. R. Soc. London* A364 (1978) 1–28.
15. M.S. Longuet-Higgins and M.J.H. Fox, Theory of the almost-highest wave (part 2): Matching and analytic extension. *J. Fluid Mech.* 85 (1978), 769–786.
16. S.J. Liao, A kind of approximate solution technique which does not depend upon small parameters (ii): an application in fluid mechanics. *Int. J. Non-Linear Mech.* 32 (1997) 815–822.
17. S.J. Liao, An analytic approximate technique for free oscillations of positively damped systems with algebraically decaying amplitude. *Int. Non-Linear Mech.* In print.
18. S.J. Liao, An explicit, totally analytic approximation of Blasius viscous flow problems. *Int. J. Non-Linear Mech.* 34 (1999) 759–778.
19. S.J. Liao, A uniformly valid analytic solution of 2D viscous flow past a semi-infinite flat plate. *J. Fluid Mech.* 385 (1999) 101–128.
20. S.J. Liao, An analytic approximation of the drag coefficient for the viscous flow past a sphere. *Int. J. Non-Linear Mech.* 37 (2002) 1–18.